Int J Theor Phys (2008) 47: 1173–1181
DOI 10.1007/s10773-007-9548-2

# Colored Cross-Correlated Noises Driven Dynamical Systems: Time Dependence of Information Entropy and Its Time Derivative

**Gurupada Goswami · Pradip Majee · Bidhan Chandra Bag**

Received: 30 May 2007 / Accepted: 28 August 2007 / Published online: 22 September 2007


**Abstract** In this paper we have studied information entropy and its time derivative for colored cross-correlated noise driven open systems. These quantities are calculated based on the Fokker–Planck description of stochastic processes. Our results consider the effect of the noise correlation strength and correlation time due to the correlation between additive and multiplicative white noises on the information entropy as well as relaxation time. The interplay of deterministic and random forces reveals non-monotonic behavior in the variation of the information entropy with damping constant.

**Keywords** Stochastic process · Colored cross-correlated noises · Information entropy · Fokker–Planck description

Study of noise driven dynamical system is always an intriguing issue in physics since random processes are ubiquitous in almost all disciplines of natural sciences such as physics, chemistry, biology [1–6]. An appropriate tool for the study of stationary and non-stationary states [7–16] in stochastic processes is Shanon's information measure [17, 18]

$$S = -\int W(q,t)\ln W(q,t)dq, \tag{1}$$

which typically is not a conserved quantity. Here $W(q,t)$ is the continuous probability distribution. $S$ as defined in the above equation is called information entropy. The time derivative of information entropy $(dS/dt)$ is related to another very important quantity, Fisher's information measure [19]. Recent studies [7–12, 19–21] show that these quantities are vital to the study of the stochastic process in detail. The $dS/dt$ has been studied in the context of Brownian motion in the presence of temperature gradient and thermodynamics beyond local equilibrium in Refs. [22, 23]. It has been also studied even in the social science (the majority-vote model) [15]. There [15] authors showed that the time derivative of information entropy exhibits a singularity at the critical point. In Ref. [16] authors has been studied

G. Goswami · P. Majee · B.C. Bag (✉)
Department of Chemistry, Visva-Bharati, Santiniketan 731 235, India
e-mail: pcbcb@rediffmail.com

the transition from the slow wave sleep to the rapid-eye-movement sleep in terms of information entropy. Thus the study of information entropy and its time derivative becomes a focal theme in the field of stochastic process. The time evolution of $S$ mainly considers the signature of the rate of phase space expansion and contraction in the stochastic process. This implies that the specific nature of the random process has a strong role to play with $S$. Generally we consider that the noise driven system is thermodynamically closed, which means that the noise of the medium is of internal origin so that the dissipation and fluctuation get related through the fluctuation-dissipation relation. However, in a number of situations the system is thermodynamically open, i.e., the dissipation and the random force are not related through fluctuation-dissipation relation [24]. In general, origins of the noise in the open systems which exert two or more random forces are different. The barrier crossing dynamics with multiplicative and additive white noises aroused strong interest in the early eighties. However, noise in some stochastic processes may have a common origin and thus can be correlated [25, 26]. Also Madureira et al. pointed out that fluctuations in some of the model parameters lead to noise contribution of both additive and multiplicative character, and are also not independent [27]. The effect of correlations between additive and multiplicative noise, either on a stationary state or on dynamics of the bistable potential system, have been widely studied [25, 26, 28–33]. In the present paper we have studied a related issue. Based on the approximate Fokker–Planck description we have calculated information entropy and its time derivative for colored cross correlated noises driven system. It is an extension of our previous works [11, 12] where we considered white cross correlation among the additive and multiplicative noises. However, from the knowledge of time derivative of information entropy one can explain experimental results in a simple way. For example, if $dS/dt$ is large then the barrier crossing rate would be high. Now we mention the relevance of the present problem in the natural processes. In a complex system there is both thermal and non-thermal environment. For example, a triggered biological system interacts with its surrounding biomolecules (non-thermal environment) as well as solvent particles (thermal environments). Effect of two environments can be represented by two noise terms in the Langevin equation of motion. Since the non-thermal environment is affected by the thermal environment, the two noises must be cross correlated. Thus in addition to the above mentioned physical processes the effect of cross correlation might be important in many biological processes such as protein folding kinetics, transport of protein, etc.

To begin with, we consider a stochastic process where both multiplicative and additive noises are present. The Langevin equation of motion for the present problem can be written as

$$\frac{dq}{dt} = -\frac{V'(q)}{\gamma} + \frac{q}{\gamma}\zeta(t) + \frac{1}{\gamma}\eta(t), \tag{2}$$

where $V'(q)$ is the derivative of potential energy expressed as a function of the particle coordinate $q$. $\gamma$ in (2) is the dissipation parameter. $\zeta(t)$ and $\eta(t)$ are white noises. The two noise terms are characterized by their means and variances as

$$\langle\zeta(t)\rangle = \langle\eta(t)\rangle = 0, \tag{3}$$

$$\langle\zeta(t)\zeta(t')\rangle = 2D\delta(t-t'), \tag{4}$$

and

$$\langle\eta(t)\eta(t')\rangle = 2D'\delta(t-t'). \tag{5}$$

Here $D$ and $D'$ are intensities of multiplicative and additive noises. Since the interaction with non-thermal environment is relatively stronger than the thermal environment, multiplicative and additive noises in (2) are due to non-thermal and thermal environments respectively. We assume that the correlation time of the $\zeta(t)$ and $\eta(t)$ are non-zero [31–35].

$$\langle\zeta(t)\eta(t')\rangle = \langle\eta(t)\zeta(t')\rangle = \frac{\lambda\sqrt{DD'}}{\tau}\exp\left(-\frac{|t-t'|}{\tau}\right), \tag{6}$$

where $\tau$ is the correlation time of the coupling between multiplicative and additive noises. By colored cross correlation we mean that the non-thermal environment is perturbed by certain range of thermal bath modes. $\lambda$ in the above equation corresponds to the correlation strength. In the limit $\tau \to 0$ (6) becomes

$$\langle\zeta(t)\eta(t')\rangle = \langle\eta(t)\zeta(t')\rangle = 2\lambda\sqrt{DD'}\delta(t-t'). \tag{7}$$

Thus the present model is very general from the point of view of cross correlation between two noises.

A general equation satisfied by the probability distribution of the (2) with (3–6) is given by [36]

$$\frac{\partial}{\partial t}\rho(q,t) = \frac{\partial}{\partial q}\frac{V'(q)}{\gamma}\rho(q,t) - \frac{\partial}{\partial q}\frac{q}{\gamma}\times\langle\zeta(t)\delta[q(t)-q]\rangle - \frac{\partial}{\partial q}\frac{1}{\gamma}\langle\eta(t)\delta[q(t)-q]\rangle, \tag{8}$$

where $\rho(q,t) = \langle\delta[q(t)-q]\rangle$; the average (8) can be calculated for Gaussian noise $\zeta(t)$ and $\eta(t)$ by the Novikov theorem [37]. Following Ref. [32] one can write an approximate Fokker–Planck equation corresponding to the Langevin equation (2) as

$$\frac{\partial\rho}{\partial t} = \left[\frac{\partial}{\partial q}\frac{V'(q)}{\gamma} - \frac{\partial}{\partial q}\left(g(q)\frac{\partial g(q)}{\partial q}\right) + \frac{\partial^2 g(q)^2}{\partial q^2}\right]\rho, \tag{9}$$

where

$$g(q) = \frac{[D' + \frac{2\lambda\sqrt{DD'}}{1+\tau V''(q_s)}q + Dq^2]^{1/2}}{\gamma}. \tag{10}$$

$q_s$ is the equilibrium position in absence of noise. It is to be noted here that the above Fokker–Planck description is valid if the following condition

$$1 + \tau V''(q_s) > 0 \tag{11}$$

is hold. For details we refer to Ref. [32]. In the limit $\tau = 0$ the Fokker–Planck description (9) is an exact description of the stochastic process [28]. The above Fokker–Planck equation can be written in the form

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial q}\frac{V'(q)\rho}{\gamma} + \frac{\partial lq\rho}{\partial q} + \frac{\partial l_1\rho}{\partial q} + Q\frac{\partial^2\rho}{\partial q^2} - \frac{2D\rho}{\gamma^2} \tag{12}$$

with,

$$l = \frac{3D}{\gamma^2}, \tag{13}$$

$$l_1 = \frac{3\lambda\sqrt{DD'}}{\gamma^2(1+2\tau)}, \tag{14}$$

and

$$Q=\frac{D'+\frac{2\lambda\sqrt{DD'}}{1+\tau V''(q_s)}q+Dq^2}{\gamma^2}. \tag{15}$$

Now multiplying $\exp(2Dt/\gamma^2)$ on both sides of the Fokker–Planck equation (12) followed by the transformation

$$W(q,t)=\rho(q,t)\exp\left(\frac{2Dt}{\gamma^2}\right), \tag{16}$$

we get

$$\frac{\partial W}{\partial t}=\frac{\partial}{\partial q}\frac{V'(q)W}{\gamma}+\frac{\partial lqW}{\partial q}+\frac{\partial l_1W}{\partial q}+Q\frac{\partial^2 W}{\partial q^2}. \tag{17}$$

Equation (16) implies that $W(q,t)$ is not normalized if $\rho(q,t)$ is a normalized probability distribution function. It also implies that a factor $\exp(-2Dt/\gamma^2)$ has to be multiplied with $\rho(q,t)\exp(2Dt/\gamma^2)$ for $W(q,t)$ to be normalized. Hence normalized $W(q,t)$ and $\rho(q,t)$ are same. For mathematical convenience we will use (17) and $W(q,t)$ for further calculations. Equation (17) can be rearranged as

$$\frac{\partial W}{\partial t}=-\frac{\partial FW}{\partial q}+Q\frac{\partial^2 W}{\partial q^2}, \tag{18}$$

where

$$F=-\Gamma q-l_1 \tag{19}$$

and

$$\Gamma=\frac{V'(q)}{q\gamma}+l. \tag{20}$$

The Fokker–Planck equation (18) can be rearranged into the general form of continuity equation

$$\frac{\partial W(q,t)}{\partial t}=-\frac{\partial j}{\partial q}, \tag{21}$$

where the current $j$ is defined as

$$j=FW-Q\frac{\partial W}{\partial q}. \tag{22}$$

We shall now calculate the time derivative of information entropy using (1) and (21). The time evolution equation for $S$ can be written as

$$\frac{dS}{dt}=\int dq\frac{\partial j}{\partial q}\ln W. \tag{23}$$

Performing partial integration on the right hand side of the above equation and then putting natural boundary conditions, $j|_{\text{boundary}}=0$, and $j\ln W|_{\text{boundary}}=0$, one obtains

$$\frac{dS}{dt}=-\int dq\frac{1}{W}j\frac{\partial W}{\partial q}. \tag{24}$$

To find the explicit time dependence of the above quantity we consider a simple external force field. In this context we choose

$$V(q)=\frac{1}{2}\omega^2q^2 \tag{25}$$

in (2) as the potential energy of a simple harmonic oscillator having frequency $\omega$. The condition (11) implies that the Fokker–Planck equation (9) is valid for arbitrary cross-correlation time $\tau$ for harmonic oscillator. For this linear stochastic process we then search for the Green's function or conditional probability solution [38–40] for the system at $q$ at time $t$ given that it had the value at $U'$ at $t=0$. This initial condition may be represented by the $\delta$-function

$$\delta(q-q')=\lim_{\epsilon\to\infty}\sqrt{\frac{\epsilon}{\pi}}\exp[-\epsilon(q-q')^2]. \tag{26}$$

$\sqrt{\epsilon/\pi}$ is the normalization constant. We now look for a solution of (18) of the form

$$W(q,t|q',0)=\exp[G(t)], \tag{27}$$

where $G(t)=-(q-\alpha(t))^2/\sigma(t)+\ln\nu(t)$.

We will see that by suitable choice of $\alpha(t),\sigma(t),\nu(t)$ one can solve (18) subject to the initial condition

$$W(q,0|q',0)=\lim_{\epsilon\to\infty}\sqrt{\frac{\epsilon}{\pi}}\exp[-\epsilon(q-q')^2]. \tag{28}$$

Comparing (27) with (28) and $G(0)$ we have $\sigma(0)=1/\epsilon$, $\alpha(0)=q'$, $\nu(0)=\sqrt{\epsilon/\pi}$. If we put (27) in (18) and equate the coefficients of equal powers of $q$ we obtain after some algebra

$$\dot\sigma(t)=-2\Gamma\sigma(t)+4Q, \tag{29}$$

$$\dot\alpha(t)=-\Gamma\alpha(t)-l_1, \tag{30}$$

and

$$\frac{1}{\nu(t)}\dot\nu(t)=-\frac{1}{2\sigma(t)}\dot\sigma(t). \tag{31}$$

Here it is to be noted that in the above calculation we have used approximate values of $q$ and $q^2$ in the diffusion coefficient $Q$ as $\langle q\rangle_{\rm eq}$ and $\langle q^2\rangle_{\rm eq}$ respectively. $\langle\ldots\rangle_{\rm eq}$ means the average value at equilibrium. Now we consider the relevant solutions of $\sigma(t)$ and $\alpha(t)$ for the present problem which satisfy the above initial conditions and are given by

$$\sigma(t)=\frac{2Q}{\Gamma}[1-\exp(-2\Gamma t)]+\sigma(0)\exp(-2\Gamma t), \tag{32}$$

and

$$\alpha(t)=\left(\alpha(0)+\frac{l_1}{\Gamma}\right)\exp(-\Gamma t)-\frac{l_1}{\Gamma}. \tag{33}$$

Now making use of (27) and (32) in (24) we finally obtain the explicit time dependence of $dS/dt$ as

$$\frac{dS}{dt}=-\Gamma+\frac{2Q}{\sigma}. \tag{34}$$

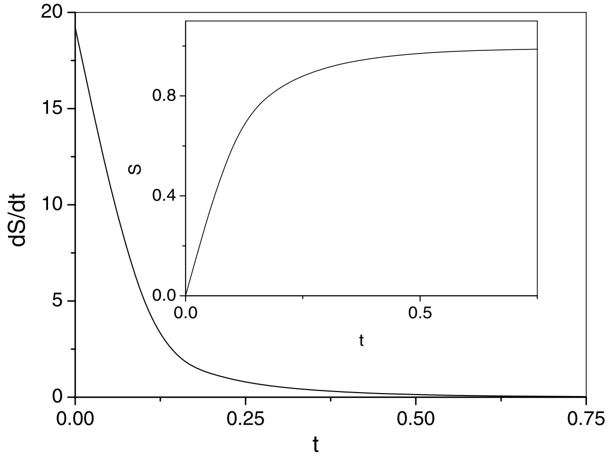


**Fig. 1** Plot of $dS/dt$ vs. time using (34) for the parameter set $\gamma = 1.0$, $\sigma(0) = 0.1$, $\alpha(0) = 1.0$, $D = D' = \omega = 1.0$, $\tau = 0.0$ and $\lambda = 0.5$. In inset plot of $S$ vs. time for the same parameter set as in the main figure

The above equation can be written in the following form

$$dS = -\Gamma dt + \frac{2Q}{\sigma} dt. \tag{35}$$

Integrating (35) we have

$$S(t) = S(0) + \frac{1}{2} \ln\left(\frac{\sigma(t)}{\sigma(0)}\right). \tag{36}$$

Now we explore how these quantities change in time. We calculate $dS/dt$ at different times and plot in Fig. 1. It shows that the time derivative of the information entropy decreases monotonically as the system approaches the stationary state. This is due to the fact that in very short time the motion of the particle is mainly governed by the deterministic force and gradually random force becomes effective, i.e., the random force has maximum tendency to expand the configuration space against the deterministic one at $t \to 0$ and it reduces with progress of time. Finally, they balance each other at equilibrium. Thus the rate of change in width of distribution function and entropy decreases regularly. But the width of the distribution function as well as the entropy increase monotonically in time and finally $S$ attains a limiting value at long time by virtue of the interplay between the random and deterministic forces. The change of $S$ in time has been shown in the inset of Fig. 1. Now we check explicitly the asymptotic behavior of both $S$ and $dS/dt$. At $t \to \infty$ (32) reduces to

$$\sigma(\infty) = \frac{2Q}{\Gamma}. \tag{37}$$

Equation (37) implies that at $t \to \infty$ the right hand side of (34) vanishes and $S$ in (36) becomes constant. These results are consistent with our natural demand. Now we consider the relaxation time. Equations (32–33) show that the relaxation time increases through increase of damping constant ($\gamma$), because $\Gamma$ decreases here. $\Gamma$, on the other hand, rises for increase of system frequency or strength of multiplicative noise and as a result of that, the non-equilibrium system relaxes more rapidly with increase of $\omega$ or $D$.

We examine now how the entropy varies with noise correlation strength $\lambda$. In Fig. 2 we have plotted $S$ vs. $\lambda$. It shows that $S$ decreases monotonically with $\lambda$. This happens because the effective noise strength ($Q$) decreases through increase of $\lambda$ and therefore width

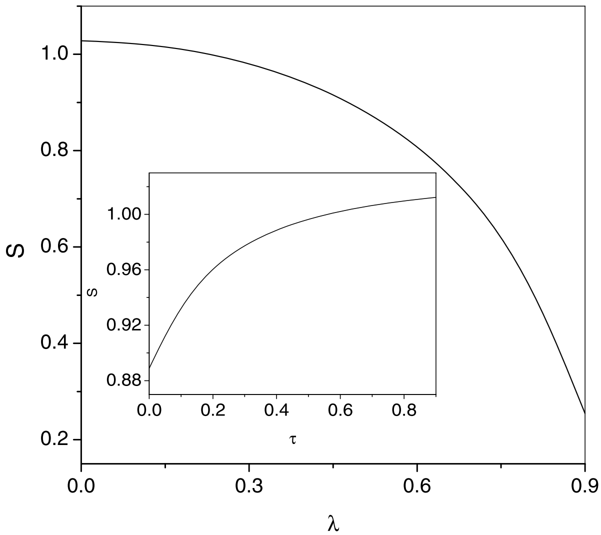


**Fig. 2** Plot of $S$ vs. strength of cross-correlation($\lambda$) using (36) for the parameter set $\gamma = 1.0$, $\sigma(0) = 0.1$, $\alpha(0) = 1.0$, $D = D' = \omega = 1.0$, $\tau = 0.0$ and $t = 0.25$. In inset plot of $S$ vs. $\tau$ for the same parameter set as in the main figure and $\lambda = 0.5$

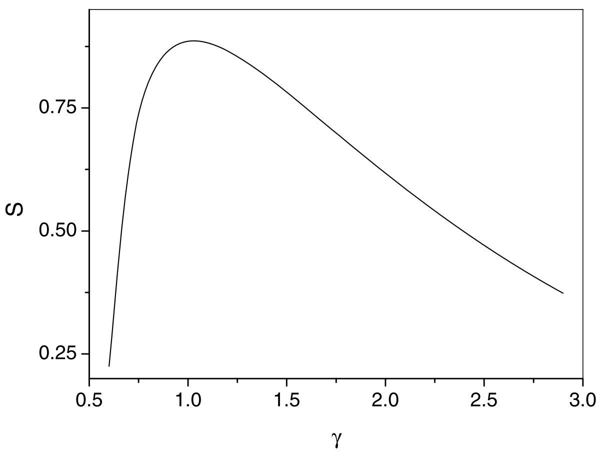


**Fig. 3** Plot of $S$ vs. $(\gamma)$ using (36) for the same parameter as in the Fig. 2

of distribution function as well as entropy become smaller as $\lambda$ rises. The opposite behavior is observed in the plot of $S$ vs. $\tau$ as $Q$ increases with increase of $\tau$. This is shown in the inset of Fig. 2.

In the next step we explore the role of damping constant $(\gamma)$ on the information entropy. $\gamma$ affects both the deterministic force and diffusion constant through $\Gamma$, $l_1$ and $Q$ respectively. The role of former (deterministic force) is opposite to the latter (diffusion constant) in regard to the expansion of configuration space. The interplay among damping constant, noise strength and system frequency is reflected in the fact that the $S$ first increases with increase of $\gamma$ and then decreases after passing through a maximum. It has been shown in Fig. 3. Thus the damping constant has an important role in the persistence of non-equilibrium state.

To examine the role of multiplicative noise on expansion of the configuration space we now plot $S$ vs. $D$ in Fig. 4. It shows that the information entropy decreases monotonically with increase of the strength of multiplicative noise. This is due to the following fact. The effective noise strength $(Q)$ as well as the effective damping constant $\Gamma$ rise for the increase

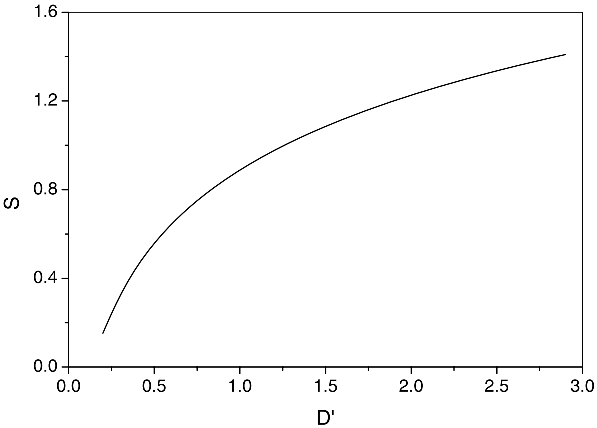


**Fig. 4** Plot of $S$ vs. strength of multiplicative noise ($D$) using (36) for the same parameter set as in Fig. 2 parameter

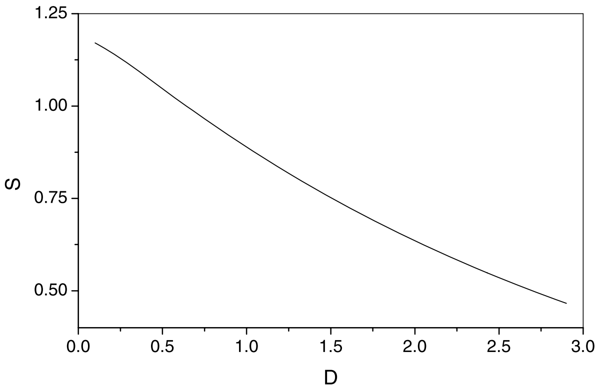


**Fig. 5** Plot of $S$ vs. strength of additive noise ($D$) using (36) for the same parameter set as in Fig. 2

of $D$. The latter dominates over the former and therefore $S$ decreases through increase of multiplicative noise strength. But the information entropy becomes larger for higher magnitudes of additive noise strength. This has been shown in Fig. 5. $\Gamma$ does not depend on the strength of additive noise but $Q$ grows with increase of $D'$. As a result, the width of the distribution function as well as the information entropy rise for the increase of strength of additive noise.

In conclusions, we have studied here the non-stationary states of a noise driven dynamical system in terms of information entropy, when the coupling between additive and multiplicative noises is colored with noise correlation time $\tau$, based on the Fokker–Planck description of stochastic processes. We consider time evolution of the information entropy and its time derivative. For demonstration we have calculated these quantities for harmonic oscillator (HO) which is an important system insights usually have a wide impact, as the HO constitutes much more than a mere example. Our main observations include the following points.

(1) The time derivative of the information entropy ($S$) monotonically decreases to zero in time but the entropy itself regularly increases to a limiting value. The relaxation time increases through increase of damping constant and it decreases for the increase in system frequency as well as the strength of multiplicative noise. The strength and the time of cross correlation given by (6) have no effect on the relaxation time.
(2) $S$ decreases with increase in strength of cross-correlation $\lambda$.

(3) The information entropy grows with increase of cross-correlation time.
(4) The interplay among damping constant $\gamma$, system frequency and random forces shows that $S$ first increases through increase of $\gamma$ reaches a maximum and finally decreases.
(5) $S$ decreases with increase of strength of the multiplicative noise. It becomes larger for higher values of additive noise strength.

The phenomena of noise driven dynamical system are governed by the phase space contraction and expansion rates. For example, the barrier crossing rate can be accelerated increasing the expansion rate. Thus we hope that the study of information entropy and its time derivative in the present paper will be useful for the understanding of the various phenomena in cross-correlated noise driven systems.

## References


1. Risken, H.: The Fokker–Planck Equation. Springer, Berlin (1989)
2. Hänggi, P., Talkner, P., Borkovec, M.: Rev. Mod. Phys. **62**, 251 (1990)
3. Mel'nikov, V.I.: Phys. Rep. **209**, 1 (1991)
4. Weiss, U.: Quantum Dissipative Systems. World Scientific, Singapore (1993)
5. Proceeding of the Second International Workshop on Fluctuations in Physics and Biology. In: Bulsara, A., et al. (eds.) Nuovo Cimento D, vol. 17 (1995)
6. Render, S.: A Guide to First Passage Processes. Cambridge University Press, Cambridge (2001)
7. Frieden, B.R.: Physics from Fisher Information. Cambridge University Press, Cambridge (1998)
8. Frieden, B.R., Plastino, A.R., Sofer, B.H.: Phys. Rev. E **60**, 48 (1999)
9. Frieden, B.R., Plastino, A., Plastino, A.R., Sofer, B.H.: Phys. Rev. E **66**, 046128 (2002)
10. Bag, B.C., Chaudhuri, J.R., Ray, D.S.: J. Phys. A: Math. Gen. **33**, 8331 (2000)
11. Bag, B.C., Banik, S.K., Ray, D.S.: Phys. Rev. E **64**, 026110 (2001)
12. Bag, B.C.: Phys. Rev. E **66**, 026122 (2002)
13. Gonzaler, D.V., Mayorga, M., Orozco, J., Salazar, L.R.: J. Chem. Phys. **118**, 6989 (2003)
14. Ai, B.Q., Wang, X.J., Liu, G.T., Liu, L.G.: Phys. Rev. E **67**, 022903 (2003)
15. Crochik, L., Tome, T.: Phys. Rev. E **72**, 057103 (2005)
16. Steyn-Ross, M.L., Steyn-Ross, D.A., Sleigh, J.W., Wilson, M.T., Wilcocks, L.C.: Phys. Rev. E **72**, 061910 (2005)
17. Jaynes, E.T.: In: Ford, W.K. (ed.) Statistical Physics. Benjamin, New York (1963)
18. Katz, A.: Statistical Mechanics. Freemann, San Francisco (1967)
19. Nikolov, B., Frieden, B.R.: Phys. Rev. E **49**, 2644 (1994)
20. Grandy, W.: Foundation of Statistical Mechanics. Reidel, Dordrecht (1987)
21. Dames, D., Nicolis, G.: Phys. Rev. E **59**, 4000 (1999)
22. Madrid, A.P., Rubi, J.M., Mazur, P.: Physica A **212**, 231 (1994)
23. Vilar, J.M.G., Rubi, J.M.: Proc. Natl. Acad. Sci. USA **98**, 11081 (2001)
24. Horsthemke, W., Lefever, R.: Noise-Induced Transitions. Springer, Berlin (1984)
25. Fedchenia, I.I.: J. Stat. Phys. **52**, 1005 (1998)
26. Fulinski, A., Telejko, T.: Phys. Lett. A **152**, 11 (1991)
27. Madureira, A.J.R., Hänggi, P., Wio, H.S.: Phys. Lett. A **217**, 248 (1996)
28. Wu, D., Cao, L., Ke, S.: Phys. Rev. E **50**, 2496 (1994)
29. Jia, Y., Li, J.-R.: Phys. Rev. Lett. **78**, 994 (1997)
30. Xie, C.W., Mei, D.C., Cao, L., Wu, D.J.: Eur. Phys. J. B **33**, 83 (2003)
31. Cao, L., Wu, D.: Phys. Rev. E **63**, 7478 (2000)
32. Jia, Y., Li, J.-R.: Phys. Rev. E **53**, 5786 (1996)
33. Mei, D., Wu, D., Cao, L.: Phys. Rev. E **59**, 3880 (1999)
34. Majee, P., Bag, B.C.: J. Phys. A: Math. Gen. **37**, 3352 (2004)
35. Luo, X., Zhu, S.: Phys. Rev. E **67**, 021104 (2003)
36. Sancho, J.M., San Miguel, M., Katz, S.L., Gunton, J.P.: Phys. Rev. A **26**, 1589 (1982)
37. Novikov, E.A.: Zh. Eksp. Teor. Fiz. **47**, 1919 (1964); Sov. Phys. JETP **29**, 1290 (1965)
38. Uhlenbech, G.E., Ornstein, L.S.: Phys. Rev. **36**, 823 (1930)
39. Chandrasekhar, S.: Rev. Mod. Phys. **15**, 1 (1943)
40. Louisell, W.H.: Quantum Statistical Properties of Radiation. Wiley, New York (1990)